Partitions of Unity

Tyrone Cutler

June 18, 2020

Contents

1 Families of Subsets

Definition 1 *A collection $\mathcal{S}=\{S_i\}_{i\in\mathcal{I}}$ of subsets $S_i\subseteq X$ of a space X, is said to* ***cover*** *X if $\bigcup_{i\in\mathcal{I}}S_i=X$. A cover $\mathcal{S}$ is said to be* ***open (closed)*** *if each S_i is open (closed). If $\mathcal{S}'=\{S'_j\}_{j\in\mathcal{J}}$ is a second cover, then we say that;*

1) $\mathcal{S}'$ is a ***subcover*** *of $\mathcal{S}$ if $\mathcal{J}\subseteq\mathcal{I}$ and $S'_j=S_j$ for each $j\in\mathcal{J}$.*

2) $\mathcal{S}'$ is a ***refinement*** *of $\mathcal{S}$ if for each $S'_j\in\mathcal{S}'$ there exists $S_i\in\mathcal{S}$ with $S'_j\subseteq S_i$. If $\mathcal{I}=\mathcal{J}$, then we say that the refinement $\mathcal{S}'$ of $\mathcal{S}$ is* ***precise****.*

3) $\mathcal{S}'$ is a ***shrinking*** *of $\mathcal{S}$ if $\mathcal{J}=\mathcal{I}$ and for each $S'_j\in\mathcal{S}'$ it holds that $\overline{S}'_j\subseteq S_j$.* $\square$

Note that a subcover is a refinement, and a shrinking is a refinement. A space is compact if and only if every open cover has a finite subcover if and only if every open cover has a finite refinement.

Definition 2 *A collection of subsets $\mathcal{S}=\{S_i\}_{i\in\mathcal{I}}$ of a space X is said to be;*

1) ***discrete*** *if each point $x\in X$ has a neighbourhood intersecting at most one of the sets in $\mathcal{S}$.*

2) ***locally-finite*** *if each point $x\in X$ has a neighbourhood intersecting at most finitely many of the sets in $\mathcal{S}$.*

3) ***point-finite*** *if each point $x \in X$ is contained in at most finitely many of the sets in $\mathcal{S}$.* $\Box$

Clearly a discrete or finite family of subsets is locally-finite, and a locally-finite family is point-finite. A subfamily of a discrete/locally-finite/point-finite family is itself discrete/locally-finite/point-finite.

Lemma 1.1 *Let X be a space and $\mathcal{S} = \{S_i\}_{\mathcal{I}}$ a family of subsets $S_i \subseteq X$. Then the following statements hold.*

1) If $\mathcal{S}$ is locally-finite, then $\overline{\bigcup_{\mathcal{I}} S_i} = \bigcup_{\mathcal{I}} \overline{S}_i$.

2) If $\mathcal{S}$ is locally-finite and each S_i is closed, then $\bigcup_{\mathcal{I}} S_i$ is closed. If each S_i is both open and closed, then $\bigcup_{\mathcal{I}} S_i$ is both open and closed.

3) If $\mathcal{S}$ is locally-finite (discrete), then the family $\{\overline{S}_i\}_{\mathcal{I}}$ is also locally-finite (discrete).

4) If $\mathcal{S}$ is locally-finite and $K \subseteq X$ is compact, then K meets at most finitely many of the sets in $\mathcal{S}$.

Proof 1) Clearly $\bigcup_{\mathcal{I}} \overline{S}_i \subseteq \overline{\bigcup_{\mathcal{I}} S_i}$ since each $\overline{S}_i \subseteq \overline{\bigcup_{\mathcal{I}} S_i}$. To show the reverse inclusion let $x \in \overline{\bigcup_{\mathcal{I}} S_i}$ and choose a neighbourhood V of x meeting only finitely many of the S_i. Then given an arbitrary neighbourhood U of x, the set $U \cap V$ meets $\bigcup_{\mathcal{I}} S_i$ nontrivially by assumption, but by construction meets only finitely many of the S_i nontrivially, say $S_1, \ldots, S_n$. This implies that U meets $\bigcup_{i=1}^n S_i$, and since U was arbitrary we can conclude that $x \in \overline{\bigcup_{i=1}^n S_i}$. This now implies that

$$x \in \overline{\bigcup_{i=1}^n S_i} = \bigcup_{i=1}^n \overline{S_i} \subseteq \bigcup_{\mathcal{I}} \overline{S}_i. \tag{1.1}$$

Parts 2) and 3) now follow easily.

For part 4) we can cover K with a finite number of open sets, each of which meets at most finitely many of the sets in $\mathcal{S}$. ∎

2 Partitions of Unity

Given a family $\{t_j\}_{i \in \mathcal{J}}$ of non-negative real numbers $t_j \in [0, \infty)$ we understand their sum to be defined by the equation

$$\sum_{j \in \mathcal{J}} t_j = \sup_{E \subseteq \mathcal{J} \text{ finite}} \sum_{j \in E} t_j. \tag{2.1}$$

If $\xi : X \to \mathbb{R}$ is a continuous function we call $\xi^{-1}(0)$ its zero set and $\xi^{-1}(\mathbb{R} \setminus 0)$ its **cozero set**. We call the closure

$$Supp(\xi) = \overline{\xi^{-1}(\mathbb{R} \setminus 0)} \tag{2.2}$$

the **support** of ξ.

Definition 3 *Let X be a space. A family $\{\xi_j\}_{j\in\mathcal{J}}$ of continuous functions $\xi_j : X \to [0,1]$ is said to be a* ***partition of unity*** *if for each $x \in X$ it holds that*

$$\sum_{j\in\mathcal{J}} \xi_j(x) = 1. \tag{2.3}$$

Write

$$\Xi = \left\{\xi_j^{-1}(0,1]\right\}_{j\in\mathcal{J}} \tag{2.4}$$

for the associated family of cozero sets.

- *We say that $\{\xi_j\}_{\mathcal{J}}$ is* ***point-finite*** *if Ξ is a point finite open covering of X.*
- *We say that $\{\xi_j\}_{\mathcal{J}}$ is* ***locally-finite*** *if Ξ is a locally finite open covering of X.*

□

Definition 4 *Let X be a space. Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be an open covering of X and $\{\xi_j\}_{j\in\mathcal{J}}$ a partition of unity on X. Set*

$$\Xi = \{\xi_j^{-1}(0,1]\}_{j\in\mathcal{J}}. \tag{2.5}$$

- *We say that $\{\xi_j\}_{\mathcal{J}}$ is* ***subordinate*** *to $\mathcal{U}$ if Ξ is an open refinement of $\mathcal{U}$.*
- *We say that $\{\xi_j\}_{\mathcal{J}}$ is a* ***numeration*** *of $\mathcal{U}$ if Ξ is a locally-finite shrinking of $\mathcal{U}$.*
- *We say that $\mathcal{U}$ is* ***numerable*** *if it has a numeration.* □

□

Unraveling the definitions we have the following. The condition $\sum_{\mathcal{J}} \xi_j(x) = 1$ for each $x \in X$ implies that $\xi_j(x) = 0$ for all but countably many $j \in \mathcal{J}$. The partition of unity $\{\xi_j\}_{\mathcal{J}}$ is point-finite if for each $x \in X$ we have $\xi_j(x) = 0$ for all but finitely many $j \in \mathcal{J}$. It is locally-finite if each $x \in X$ has a neighbourhood U such that $\xi_j|_U = 0$ for all but finitely many $j \in \mathcal{J}$. If $\{\xi_j\}_{\mathcal{J}}$ is locally-finite, then according to Lemma 1.1, so is the family of supports $\{\overline{\xi_j^{-1}(0,1]}\}_{\mathcal{J}}$. If $\{\xi_j\}_{\mathcal{J}}$ is subordinated to an open cover $\mathcal{U} = \{U_i\}_{\mathcal{I}}$, then for each $j \in \mathcal{J}$ we have $\xi^{-1}(0,1] \subseteq U_i$ for some $i \in \mathcal{I}$. If $\{\xi_j\}_{\mathcal{J}}$ is a numeration of $\mathcal{U}$, then $\mathcal{I} = \mathcal{J}$, and for each $i \in \mathcal{I}$ we have $Supp(\xi_i) = \overline{\xi_i^{-1}(0,1]} \subseteq U_i$.

Before continuing we must make it clear that our definitions above are not completely standard. For starters, many authors require all partitions of unity to be locally-finite. We view this as unnecessarily restrictive, since many of the most natural things to write down often fail to be locally-finite. In particular, the algebraic methods used for constructing partitions of unity very rarely yield locally-finite families.

A second difference is that our usage of the word 'subordinate' differs from many texts, where it appears in place of our 'numerable'. In any case it is the locally finite partitions of unity which one would always prefer to work with if possible. From our perspective we will for the most part be able to sweep all references to partitions of unity under the rug, and work solely with numerable coverings, once we have established their existence. Our viewpoint, then, is that our machinery should be flexible enough to allow for a reasonably general input, and yet from it produce from it the most usable output. The rest of this section is dedicated to assembling this machinery.

Any real-valued map $\pi : X \to \mathbb{R}$ determines a non-negative function

$$\pi^+ : X \to [0, \infty), \qquad x \mapsto \max\{\pi(x), 0\}. \tag{2.6}$$

If $\{\pi_i\}_{\mathcal{I}}$ are a locally-finite family of maps $\pi_i : X \to [0, \infty)$ such that $\sum_{\mathcal{I}} \pi_i$ is positive and finite throughout X, then the collection of functions

$$\pi_i'(x) = \frac{\pi_i(x)}{\sum_{\mathcal{I}} \pi_j(x)}, \qquad i \in \mathcal{I} \tag{2.7}$$

forms a locally-finite partition of unity on X. The continuity of the π_i' follows from the assumption of local finiteness, since this implies that locally on X the sum $\sum_{\mathcal{I}} \pi_j$ has only finitely many terms.

Lemma 2.1 *Let $\{\xi_j\}_{\mathcal{J}}$ be a partition of unity on a space X and $\epsilon > 0$. Then each point $x \in X$ has a neighourhood $U_x(\epsilon) \subseteq X$ such that $\xi_j < \epsilon$ throughout $U_x(\epsilon)$ for all but finitely many $j \in \mathcal{J}$.*

Proof For fixed $x \in X$ there is a finite subset $E \subseteq \mathcal{I}$ such that $\sum_{j \in E} \xi_j(x) > 1 - \epsilon$. Since E is finite

$$U_x(\epsilon) = \left\{ y \in X \;\middle|\; \textstyle\sum_{j \in E} \xi_j(y) > 1 - \epsilon \right\} \tag{2.8}$$

is an open set containing x. If $\xi_k(x) \geq \epsilon$ then it must be that $k \in E$, since if it is not, then

$$\xi_k(x) + \sum_{j \in E} \xi_j(x) > 1 \tag{2.9}$$

which is a contradiction to the fact that the ξ_j form a partition of unity. ∎

Corollary 2.2 *If $\{\xi_j\}_{\mathcal{J}}$ is a partition of unity on a space X, then the function*

$$\mu : X \to [0, 1], \qquad x \mapsto \sup_{\mathcal{J}} \xi_j(x) \tag{2.10}$$

agrees locally with the maximum over a finite subset $E \subseteq \mathcal{J}$. In particular it is strictly positive and continuous.

Proof For $x \in X$ fix $0 < \epsilon < \mu(x)$ and let $U_x(\epsilon) \subseteq X$ be as in 2.1. Choose a finite $E \subseteq \mathcal{J}$ such that if $k \notin E$, then $\xi_k < \epsilon$ outside of $U_x(\epsilon)$. Then on $U_x(\epsilon)$ the function μ agrees with the continuous function $y \mapsto \max_E \xi_j(y)$. ∎

Theorem 2.3 *Let $\{\sigma_i\}_{i \in \mathcal{I}}$ be a partition of unity on a space X. Then there exists a locally-finite partition of unity $\{\xi_i\}_{i \in \mathcal{I}}$ on X indexed by the same set such that for each $i \in \mathcal{I}$ it holds that*

$$Supp(\xi_i) \subseteq \sigma_i^{-1}(0, 1]. \tag{2.11}$$

In particular the open covering $\{\sigma_i^{-1}(0, 1]\}_{\mathcal{I}}$ is numerable.

Proof Let $\mu = \sup \sigma_i$ be as in 2.2. Then for each $i \in \mathcal{I}$ function $\widetilde{\xi_i} : X \to I$ give by

$$\widetilde{\xi_i}(x) = \max\{0, \sigma_i(x) - \mu(x)/2\} \tag{2.12}$$

is continuous. Fix $x_0 \in X$ and set $\epsilon = \mu(x_0)/4$. Then there exists a neighbourhood U of x_0 and a finite set $E \subseteq \mathcal{I}$ such that $\sigma_k(x) < \epsilon < \mu(x)/2$ for all $x \in U$ and $k \notin E$. This implies that $\widetilde{\xi_k}(x) = 0$ for all $x \in U$, $k \notin E$, and shows that the $\widetilde{\xi_i}$ form a locally-finite collection.

Now for each $x \in X$, the sum $\sum_{\mathcal{I}} \widetilde{\xi_i}(x)$ is bounded above by $\sum_{\mathcal{I}} \xi_i(x) = 1$. On the other hand there exists an index $i \in \mathcal{I}$ such that $\mu(x) = \sigma_i(x)$, and in particular $\sum_{\mathcal{I}} \widetilde{\xi_i}(x) \neq 0$. Thus we can normalise the $\widetilde{\xi_i}$ as in (2.7) to get a locally-finite partition of unity $\{\xi_i\}_{i \in \mathcal{I}}$, where

$$\xi_i = \frac{\widetilde{\xi_i}}{\sum_{\mathcal{I}} \widetilde{\xi_j}}, \qquad i \in \mathcal{I}. \tag{2.13}$$

To see that it satisfies the required condition we check that

$$\begin{aligned} \overline{\xi_i^{-1}(0,1]} &= \overline{\widetilde{\xi}_i^{-1}(0,1]} \\ &\subseteq \{x \in X \mid \sigma_i(x) \geq \mu(x)/2\} \\ &\subseteq \sigma_i^{-1}(0,1]. \qquad \blacksquare \end{aligned} \tag{2.14}$$

Corollary 2.4 *Let $\mathcal{U}$ be an open covering of a space X. Assume that $\mathcal{U} = \{U_i\}_{\mathcal{I}}$ has a subordinated partition of unity. Then $\mathcal{U}$ is numerable.*

Proof Given a partition of unity subordinate to $\mathcal{U}$ we can use Theorem 2.3 to find a numeration $\{\xi_a\}_{\mathcal{A}}$ of its cozero sets. The family of supports of $\{\xi_a\}_{\mathcal{A}}$ then forms a closed locally-finite refinement of $\mathcal{U}$. Now, for each $a \in \mathcal{A}$ choose $i(a) \in \mathcal{I}$ such that $Supp(\xi_a) \subseteq U_{i(a)}$. Then for each $i \in \mathcal{I}$ put

$$\pi_i = \sum_{i(a)=i} \xi_a \tag{2.15}$$

with the understanding that the empty sum returns the zero function. Since the supports of the ξ_a are locally-finite, the π_i are continuous. Moreover, for each $x \in X$ and $i \in \mathcal{I}$ we have

$$0 \leq \pi_i(x) \leq \sum_{\mathcal{A}} \xi_a(x) = 1. \tag{2.16}$$

Finally, since the ξ_a are locally finite we find

$$\begin{aligned} Supp(\pi_i) &= \overline{\pi_i^{-1}(0,1]} \\ &= \overline{\bigcup_{i(a)=i} \xi_a^{-1}(0,1]} \\ &= \bigcup_{i(a)=i} \overline{\xi_a^{-1}(0,1]} \\ &= \bigcup_{i(a)=i} Supp(\xi_a) \\ &\subseteq U_i. \end{aligned} \tag{2.17}$$

It remains to show that the family $\{\pi_i\}_{\mathcal{I}}$ is locally-finite. So let $x \in X$ and choose a neighbourhood W and a finite subset $E \subseteq \mathcal{A}$ such that $Supp(\xi_b) \cap W = \emptyset$ if $b \notin E$. Then if $Supp(\pi_i) \cap W \neq \emptyset$, then it must be that $i = i(b)$ for some $b \in E$. Hence there can be only finitely many π_i for which this intersection is nonempty. $\blacksquare$

2.1 Paracompact Spaces and Partitions of Unity

The purpose of this section is to discuss the rôle that partitions of unity play in the in relation to *paracompactness*. We will not need to discuss or understand paracompact spaces deeply for our work, but since we make mention to the concept we have included some accompanying notes. In these notes a full proof of the following theorem is given. Since it offers a complete characterisation of paracompact spaces, the reader may take its statement as a definition for *paracompactness*.

Theorem 2.5 (Theorem/Definition) *Let X be a T_1-space. Then the following are equivalent.*

1) X is paracompact.

2) Each open cover of X is numerable.

3) Each open cover of X has a partition of unity subordinated to it. $\blacksquare$

Examples of paracompact spaces are plentiful. For example Stone proved the following celebrated result, whose proof can be found in [4].

Theorem 2.6 (Stone) *Every metrisable space is paracompact.* $\blacksquare$

This theorem alone has many useful applications. For example, the Urysohn Metrisation Theorem states that a second-countable completely regular space is metrisable, so a consequence of 2.6 is the following.

Theorem 2.7 *Every second-countable[1] (topological) manifold is paracompact.* $\blacksquare$

More directly useful to this course is following fact.

Theorem 2.8 *Every CW complex is paracompact.* $\blacksquare$

A proof of this will be given in a later lecture. The impatient reader can track it down in [2]. In fact, a weaker statement follows directly from 2.6, since it is known that a CW complex is metrisable if and only if it is first-countable if and only if it is locally finite [2].

We end this section with a list of elementary statements which the reader is invited to either prove form themselves or take for granted.

1) Any compact Hausdorff space is paracompact.

2) A closed subspace of a paracompact space is paracompact. Arbitrary subspaces need not be.

[1]A space X is said to be **first-countable** if each of its points has a countable neighbourhood base. X is said to be **second-countable** if its topology has a countable base.

3) A disjoint union of arbitrarily many paracompact spaces is paracompact.

4) A product of paracompact spaces may fail to be paracompact. A product of a paracompact space and a compact Hausdorff space is paracompact.

5) If X, Y are paracompact and $f : X \to Y$ is a closed surjection, then Y is paracompact.

6) If X, Y are paracompact, $A \subseteq X$ is closed and $f : A \to Y$ is a map, then the adjunction space $Y \cup_f X$ is paracompact.

2.2 Sample Applications for Partitions of Unity

Example 2.1 Let $\mathcal{H}$ be a Hilbert space and denote by $S = S(\mathcal{H}) = \{x \in \mathcal{H} \mid \|x\|^2 = 1\}$ its unit sphere. Choose an orthonormal basis $\{e_i\}_{i\in\mathcal{I}}$ for $\mathcal{H}$. Then the functions

$$\xi_i : S(\mathcal{H}) \to \mathbb{R}, \qquad x \mapsto |\langle e_i, x\rangle|^2 \tag{2.18}$$

define a partition of unity which is in general not even point-finite. Continuity of the ξ_i is clear, and the fact that $\sum_{\mathcal{I}} \xi_i = 1$ follows from Parseval's Identity (see [5] pg. 45)

$$\|x\|^2 = \sum_{\mathcal{I}} |\langle x, e_i\rangle|^2, \qquad x \in \mathcal{H}. \tag{2.19}$$

As mentioned above, if $\mathcal{H}$ is infinite-dimensional, then the family of cozero sets $\{\xi_i^{-1}(0,1]\}_{\mathcal{I}}$ is not a locally-finite cover of $S(\mathcal{H})$. However Theorem 2.3 applies, and the covering is still numerable. □

Example 2.2 Let G be a topological group. If $U \subseteq G$ is a neighbourhood of the identity, then $\{g\cdot U\}_{g\in G}$ is a numerable open cover. This observation leads to the *Birkhoff Metrization Theorem*, which states that a topological group is metrisable if and only if it is first-countable. The details of all this are not hard and can be found on page 5 of [1]. □

Example 2.3 Let $X = (X, d)$ be a metric space and $\mathcal{U}$ a locally-finite open cover. For each $i \in \mathcal{I}$ let $\widetilde{\xi}_i : X \to [0, \infty)$ be the map

$$\widetilde{\xi}_i(x) = d(x, X \setminus U_i). \tag{2.20}$$

The sum $\sum_{\mathcal{I}} \widetilde{\xi}_i(x)$ is defined and positive throughout X, so we get a partition of unity by normalisation

$$\xi_i(x) = \frac{\widetilde{\xi}_i(x)}{\sum_{\mathcal{I}} \xi_j(x)}, \qquad i \in \mathcal{I}. \tag{2.21}$$

A direct check shows that $\{\xi_i\}_{\mathcal{I}}$ is a numeration of $\mathcal{U}$.

Of course we know that any open cover of X is numerable, but the example is to demonstrate that it is often easier to work with given data than to appeal to abstract reasoning. □

Example 2.4 Here we discuss the failure of the converse of Lemma 2.1. Let $X = [0, 3]$ and for each $n \geq 1$ define $f_n : X \to [0, \infty)$ by

$$f_n(x) = \begin{cases} n \cdot x & 0 \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq \frac{2}{n} \\ n \cdot (\frac{3}{n} - x) & \frac{2}{n} \leq x \leq \frac{3}{n} \\ 0 & \frac{3}{n} \leq x \leq 3. \end{cases} \tag{2.22}$$

Then each f_n is continuous and the family $\{f_n\}_{\mathbb{N}}$ is point-finite. Indeed, $f_n(0) = 0$ for all $n \in \mathbb{N}$, and if $x \in (0, 3]$, then $f_n(x) = 0$ whenever $n \geq \lfloor \frac{3}{x} \rfloor$. In particular the sum

$$f(x) = \sum_{\mathbb{N}} f_n(x) \tag{2.23}$$

is finite for each $x \in X$. On the other hand, $\{f_n\}_{\mathbb{N}}$ is not locally-finite, and all the f_n are non-zero on any neighbourhood of 0. This has the consequence that the function f defined by (2.23) is not continuous in x at 0.

The moral is that one must take care when attempting to obtain partitions of unity through normalisation. The example show that even point-finite families of maps can have bad behaviour. In general we can prove the following, which greatly generalises Lemma 2.1

Lemma 2.9 *Let $\{f_i : X \to [0, \infty)\}_{\mathcal{I}}$ be a family of continuous functions on a space X and assume that $\sum_{\mathcal{I}} f_i(x)$ is finite for each $x \in X$. Then the assignment $f_{\mathcal{I}} : x \mapsto \sum_{\mathcal{I}} f_i(x)$ defines a continuous function $f_{\mathcal{I}} : X \to [0, \infty)$ if and only if for each $x \in X$ there is an $\epsilon > 0$ for which there exists a neighbourhood $U \subseteq X$ of x and a finite subset $E \subseteq \mathcal{I}$, such that $\sum_{i \in \mathcal{I} \setminus E} f_i < \epsilon$ throughout U.* ∎

Example 2.5 Let M be a second-countable C^r manifold, $0 \leq r \leq \infty$. Then M is paracompact, and hence each open cover $\mathcal{U}$ of M is numerable. However, it is true in this case that a numeration $\{\xi_i\}_{\mathcal{I}}$ can be constructed for $\mathcal{U}$ such that each function

$$\xi_i : M \to [0, 1] \subseteq \mathbb{R} \tag{2.24}$$

is of smoothness class C^r. See Hirsch [3] pg. 43 for the construction. The existence of C^r partitions of unity leads directly to the famous *Whiney Embedding Theorems* [3] pg. 24. The first step to proving these theorem is indicated below.

Proposition 2.10 *Let M be a smooth compact manifold of dimension n. Then M can be embedded in $\mathbb{R}^N$ for some N.*

Proof Cover M by finitely many charts $(U_1, h_1), \dots, (U_k, h_k)$ and choose an enumeration $\{\xi_1, \dots, \xi_k : M \to [0, 1]\}$. For $i = 1, \dots, k$ define

$$\varphi_i : M \to \mathbb{R}^n, \qquad \varphi_i(x) = \begin{cases} \xi_i(x) \cdot h_i(x) & x \in U_i \\ 0 & x \in \setminus U_i. \end{cases} \tag{2.25}$$

Since $Supp(\xi_i) \subseteq U_i$ this is well defined and continuous. Now set $N = k + k \cdot n$ and define

$$\varphi : M \to \mathbb{R}^N, \qquad \varphi(x) = (\xi_1(x), \ldots, \xi_k(x), \varphi_1(x), \ldots, \varphi_k(x)). \tag{2.26}$$

Then φ is continuous. If $\varphi(x) = \varphi(y)$, then $\xi_i(x) = \xi_i(y)$ and $\varphi_i(x) = \varphi_i(y)$ for all $i = 1, \ldots, k$. In this case we find j such that $\xi_j(x) = \xi_j(y) \neq 0$ and conclude that $h_j(x) = h_j(y)$. Since h_j is chart, it is injective, and so we see that $x = y$. The conclusion is that φ is injective. Since M is compact and $\mathbb{R}^n$ is Hausdorff, this implies that φ is an embedding. ∎

Example 2.6 Recall the following terminology.

Definition 5 *A space X is said to be σ-**compact** if it is a union of countably many compact subsets.* □

Claim A locally compact second-countable paracompact space X is σ-compact. In particular it admits an exhaustion function.

An **exhaustion function** is a continuous map $f : X \to \mathbb{R}$ such that for each $c \in \mathbb{R}$, the sublevel set $f^{-1}(-\infty, c] \subseteq X$ is compact. Clearly the proof of the claim is established once such a function is produced, since for $n \in \mathbb{N}$ we let $K_n = f^{-1}(-\infty, n] \subseteq X$.

Proof Choose a countable open cover $\{U_n\}_{\mathbb{N}}$ of X. Since X is locally compact Hausdorff, we can choose each U_n so that $\overline{U}_n$ is compact. Assume this done and a numeration $\{\xi_n\}_{\mathbb{N}}$ for the cover. Now define $f : X \to \mathbb{R}$ by setting

$$f(x) = \sum_{n=1}^{\infty} n \cdot \xi_n(x). \tag{2.27}$$

Since the numeration is locally finite, each $x \in X$ has a neighbourhood on which only finitely many terms contribute to the sum. This implies that f is continuous. Moreover f is positive, since $f(x) \geq \sum_{\mathbb{N}} \xi_n(x) = 1$.

Now, if $x \notin \bigcup_{n=1}^{r} \overline{U}_n$, then $\xi_n(x) = 0$ for $1 \leq n \leq r$, so

$$f(x) = \sum_{n=r+1}^{\infty} n \cdot \xi_n(x) > \sum_{n=r+1}^{\infty} r \cdot \xi_n(x) = r \cdot \sum_{n=r+1}^{\infty} \xi_n(x) = r. \tag{2.28}$$

Then since the U_n cover X, the above implies that if $f(x) < r$, then $x \in \bigcup_{n=1}^{r} \overline{U}_n$. Thus if $c \in \mathbb{R}$ and r is an integer greater than c, then $f^{-1}(-\infty, c]$ is a closed subset of the compact $\bigcup_{n=1}^{r} \overline{U}_n$, and so is itself compact. ∎

Note that the theorem is applicable to all smooth or topological manifolds. □

3 An Application To Homotopy Theory

What follows below is a typical application of numerability to homotopy theory. The idea is made clearest when the numerable cover consists of just two sets. So let $U_0, U_1 \subseteq X$ be an open cover numerated by $\{\xi_0, \xi_1 : X \to I\}$. By writing $\xi_1 = 1 - \xi_0$, the information of the numeration can be compressed into the single function $\xi = \xi_0 : X \to I$. Then $Supp(\xi) \subseteq U_0$, so ξ is identically zero on $X \setminus X_0$. Moreover ξ is identically 1 outside of U_1 and has $\xi^{-1}(0,1) \subseteq U_0 \cap U_1$.

Proposition 3.1 *Let $U_0, U_1 \subseteq X$ be a numerable open cover of a space X. Suppose that Y is a space, $Y_0, Y_1 \subseteq Y$ subspaces, and that there are maps*

$$\phi_i : U_i \to Y_i, \qquad i = 0, 1. \tag{3.1}$$

Assume that $H : (U_0 \cap U_1) \times I \to Y$ is a homotopy $H : \phi_0|_{U_{01}} \simeq \phi_1|_{U_{01}}$, where $U_{01} = U_0 \cap U_1$. Then there exists a map $\phi : X \to Y$ such that $\phi(U_i) \subseteq Y_i$, and a pair of homotopies

$$\begin{aligned} F &: U_0 \times I \to Y_0 & \qquad G &: U_1 \times I \to Y_1 \\ \phi_0 &\simeq \phi|_{X_0} & \phi_1 &\simeq \phi|_{X_1}. \end{aligned} \tag{3.2}$$

Proof As discussed above we can get a numeration for $\{U_0, U_1\}$ by specifying a single map ξ which in particular must satisfy $Supp(\xi) \subseteq U_0$ and $\xi(X \setminus U_0) = 0$. We can assume without loss of generality that the homotopy H_t is independent of t on $[0, \frac{1}{4}]$ and on $[\frac{3}{4}, 1]$. If this is not already so, then we can easily construct a track homotopy to replace it with one which is. Now define $\phi : X \to Y$ by setting

$$\phi(x) = \begin{cases} H(x, \xi(x)) & x \in U_0 \cap U_1 \\ \phi_0(x) & x \in \xi^{-1}[0, \frac{1}{4}) \\ \phi_1(x) & x \in \xi^{-1}(\frac{3}{4}, 1]. \end{cases} \tag{3.3}$$

Then ϕ is well-defined and continuous. The homotopies F_t, G_t are not difficult to write down, and we will give only F_t, leaving the construction of G_t to the reader. This is given by

$$F_t(x) = \begin{cases} H(x, t \cdot \xi(x)) & x \in U_0 \cap U_1 \\ \phi_0(x) & x \in \xi^{-1}[0, \frac{1}{4}). \end{cases} \tag{3.4}$$

∎

Something the reader might like to keep in mind when trying to understand the last proposition is that if U_0, U_1 cover X, then X is the pushout in the following square

$$\begin{tikzcd} U_0 \cap U_1 \arrow[r] \arrow[d] & U_1 \arrow[d] \\ U_0 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X. \end{tikzcd} \tag{3.5}$$

As usual this means that given maps $\phi_i : U_i \to Y$, $i = 0, 1$, which agree on $U_0 \cap U_1$, it is possible to glue them together to get a globally defined map $\phi : X \to Y$.

With the added assumption of numerability it then becomes possible to perform a gluing even when ϕ_0 and ϕ_1 are only homotopic over $U_0 \cap U_1$. Of course in this case the resulting map ϕ is not unique, but there is some control over its homotopy class, which will depend on those of ϕ_0, ϕ_1 and the track homotopy class of H_t.

There is an obvious way to generalise Proposition 3.1 by allowing for numerable coverings with more than two members. An inductive argument based around 3.1 immediately leads to a statement when X is covered by any finite numbers of sets. It is even true that a statement can be made in the case of infinite coverings, but the inductive argument will not work here, and more intricate machinery is required for this.

Example 3.1 Let $j : A \hookrightarrow X$ be a closed cofibration and choose for it a Strøm structure (ϕ, H). Set $U_0 = \phi^{-1}[0,1)$ and $U_1 = X \setminus A$. Then $\mathcal{U} = \{U_0, U_1\}$ is a numerable cover of X. The pair $\{\phi, 1-\phi\}$ is a partition of unity subordinate to $\mathcal{U}$, but the functions must be perturbed to achieve a numeration. □

Example 3.2 Notice that there is a parametrised version of Proposition 3.1. If we fix a space B, assume that X, Y are spaces over B, that the maps ϕ_0, ϕ_1 are maps over B, and that H is a fibrewise homotopy, then the map ϕ obtained in 3.3 is a map over B, and F, G are fibrewise homotopies. □

4 Milnor's Theorem and the Stacking Lemma

The material of this section is needed to prove the Homotopy Theorem for Bundles. Since the lemmas presented are both of a technical nature and of independent interest, we have chosen to give their proofs at this point, and isolate them from our treatment of bundles.

Theorem 4.1 (Milnor) *Let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be a numerable open covering of a space X. Then there exists a countable, numerable open covering $\mathcal{V} = \{V_n\}_{n\geq 1}$ of X such that each V_n is a disjoint union of open sets, each of which is contained in some U_i.*

Proof Fix a numeration $\{\xi_i\}_{\mathcal{I}}$ of $\mathcal{U}$. For each finite subset $E \subseteq \mathcal{I}$ let $V(E) \subseteq X$ be the open set

$$V(E) = \{x \in X \mid \xi_i(x) < \xi_j(x) \text{ if } i \in \mathcal{I} \setminus E \text{ and } j \in E\}. \tag{4.1}$$

Note that if $E \neq F$ are finite subsets of $\mathcal{I}$ of the same cardinality, then there is a $j \in E$ with $j \notin F$, and a $j' \in F$ with $j' \notin E$, and this implies that $V(E) \cap V(E') = \emptyset$.

Now, for each $x \in X$ let $E(x)$ denote the finite set of indices $i \in \mathcal{I}$ for which $\xi_i(x) > 0$. Then $V(E(x))$ is an open set which is contained inside U_i if $i \in E(x)$. For $n \geq 1$ put

$$V_n = \bigcup_{x\in X} \bigcup_{|E(x)|=n} V(E(x)). \tag{4.2}$$

Then V_n is a disjoint union of open sets, each of which is contained in some U_i.

It's clear that the family $\{V_n\}_{n\geq 1}$ covers X, so to complete we only need to show that it is numerable. For this we proceed as follows. For a finite subset $E \subseteq \mathcal{I}$ let $\sigma_E : X \to [0, \infty)$ be the function

$$\sigma_E(x) = \min_{i\in E,\, j\in\mathcal{I}\setminus E} \{\xi_i(x) - \xi_j(x)\}. \tag{4.3}$$

Similarly, for $n \geq 1$ let $\rho_n : X \to [0,\infty)$ be the function

$$\rho_n(x) = \sum_{|E|=n} \sigma_E(x). \tag{4.4}$$

Then $V(E) = \sigma_E^{-1}((0,\infty))$ and $V_n = \rho_n^{-1}((0,\infty))$. To see that $\{\rho_n\}_{n\geq 1}$ is locally finite we need only observe that each point $x \in X$ has a neighbourhood on which all but finitely many of the ξ_i vanish. This implies that $\rho_N(x) = 0$ for sufficiently large $N \subseteq \mathcal{I}$. Thus we convert $\{\rho_n\}_{n\geq 1}$ into a numeration of $\{V_n\}_{\mathbb{N}}$ by normalising. ∎

Addendum In the statement of 4.1, if each point of X is contained in at most n members of $\mathcal{U}$, then $\mathcal{V}$ can be chosen to be finite.

Proof This is clear, since with V_k as in (4.2) the assumption gives $V_k = \emptyset$ if $k > n$. ∎

The following statement we we call the Stacking Lemma is crucial for out proof of the Homotopy Theorem. It has an obvious, but difficult to prove, generalisation which replaces I by any compact space.

Proposition 4.2 (Stacking Lemma) *Let B be a space and $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ a numerable open covering of $X \times I$. Then there exists a numerable open covering $\{V_j\}_{j\in\mathcal{J}}$ of X, and a family of real numbers $\{\epsilon_j \in (0,\infty)\}_{j\in\mathcal{J}}$ with the property that for each $j \in \mathcal{J}$ and all $s < t \in I$ with $t - s < \epsilon_j$, there exists $i \in \mathcal{I}$ such that $V_j \times [s,t] \subseteq U_i$.*

Proof Let $(\xi_i)_{\mathcal{I}}$ be a numeration of $\mathcal{U}$. Put $\mathcal{J} = \bigsqcup_{n\geq 1} \mathcal{I}^n$. Then for $\underline{j} = (i_1,\ldots,i_n) \in \mathcal{I}^n$ let $\rho_n : X \to [0,1]$ be the map

$$\rho_{\underline{j}}(b) = \prod_{k=1}^{n} \min\left\{\xi_{i_k}(b,t) \,\middle|\, t \in \left[\frac{k-1}{n+1}, \frac{k+1}{n+1}\right]\right\} \tag{4.5}$$

and set

$$V_{\underline{j}} = \rho_{\underline{j}}^{-1}\left((0,1]\right), \qquad\qquad \epsilon_{\underline{j}} = \frac{1}{2n}. \tag{4.6}$$

Then

$$V_{\underline{j}} \subseteq \bigcap_{k=1}^{n} \left\{b \in X \,\middle|\, \{b\} \times \left[\frac{k-1}{n+1}, \frac{k+1}{n+1}\right] \subseteq U_{i_k}\right\} \tag{4.7}$$

so the $\epsilon_{\underline{j}}$ satisfy the required condition. To complete we need to show that $\{V_{\underline{j}}\}_{\mathcal{J}}$ is a locally finite open covering of X.

The $V_{\underline{j}}$ are open, and to see that they cover X we work as follows. Fix a point $x \in X$. Then since $\mathcal{U}$ is an open covering of $X \times I$, for each $t \in I$ we can find an open neighbourhood $W'_t \subseteq X$ of x and an open neighbourhood $W''_t \subseteq I$ of t such that $W'_t \times W''_t$ is contained inside some U_i and meets at most finitely many other members of $\mathcal{U}$. As we vary t, the sets W''_t cover I, so by compactness we can find finitely many, say $W''_1,\ldots,W''_n$, which also do. Choose

$r \geq 1$ such that $\frac{2}{r+1}$ is a Lebesgue number[2] for this covering. Then if $W'_1, \ldots, W'_n \subseteq X$ are the corresponding neighbourhoods of x, we have that $x \in W = W'_1 \cap \cdots \cap W'_n$ and each

$$W \times \left[\frac{k-1}{r+1}, \frac{k+1}{r+1}\right], \qquad k = 1, \ldots, r \tag{4.8}$$

is contained inside some U_{i_k}. This implies that x lies in $V_{\underline{j}}$ for $\underline{j} = (i_1, \ldots, i_r)$.

Next we must show that $\{V_{\underline{j}}\}_{\mathcal{J}}$ is locally finite. With $x \in X$ still fixed, the set W of (4.8) was constructed to have the property that $W \times I$ meets only finitely many of the U_i. This implies that for each fixed n the family $\{V_{\underline{j}} \mid \underline{j} = (i_1, \ldots, i_m),\ m \leq n\}$ is locally finite. Thus for each $n \geq 1$ let $\pi_n : X \to I$ be the map

$$\pi_n(x) = \max\{\rho_{\underline{j}}(x) \mid \underline{j} = (i_1, \ldots, i_m)\ m < n\} \tag{4.9}$$

and for $\underline{j} = (i_1, \ldots, i_n)$ set

$$\widetilde{\rho}_{\underline{j}}(b) = \max\{0, \rho_{\underline{j}}(b) - n \cdot \pi_n(b)\}. \tag{4.10}$$

Then

$$\widetilde{\rho}_{\underline{j}}^{-1}((0,1]) \subseteq \rho_{\underline{j}}^{-1}((0,1]) = V_{\underline{j}}. \tag{4.11}$$

With x as above let n be the minimal integer such that $\rho_{\underline{j}}(x) > 0$ for some $\underline{j} = (i_1, \ldots, i_n)$. Then $\widetilde{\rho}_{\underline{j}}(x) = \rho_{\underline{j}}(x) > 0$. If $N > n$ is such that $N \cdot \rho_{\underline{j}}(x) > 1$, then $N \cdot \rho_{\underline{j}}(x') > 1$ for all x' in some neighbourhood V' of x. Thus if $\underline{j}' = (i_1, \ldots, i_N)$, then $\widetilde{\rho}_{\underline{j}'}$ vanishes on V'. In particular the family $\{\widetilde{\rho}_{\underline{j}}\}_{\mathcal{J}}$ is locally finite and we get a locally finite partition of unity by replacing $\widetilde{\rho}_{\underline{j}}$ with the normalised function

$$\frac{\widetilde{\rho}_{\underline{j}}}{\sum_{\mathcal{J}} \widetilde{\rho}_{\underline{k}}}. \qquad \blacksquare \tag{4.12}$$

References

[1] K Austin, J. Dydak, *Partitions of Unity and Coverings*, Top. App. **173**, (2014), 74-82.

[2] R. Fritsch, R. Piccinini, *Cellular Structures in Topology*, Cambridge University Press, (1990).

[3] M. Hirsch, *Differential Topology*, Springer-Verlag, (1976).

[4] J. Nagata, *Modern General Topology*, Second Edition, North-Holland, (1985).

[5] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, (1980).

[2] i.e. any open ball of radius $< \frac{2}{r+1}$ is contained inside some W_k.